ECE 536 – Spring 2022

Homework #4 – Solutions

Problem 1)

If we first consider the photon field with energy E_{21}, we can write our usual expression for the density of states

$$\begin{aligned} N(E_{21}) &= \frac{2}{V}\sum_k \delta(E_2 - E_1 - E_k) \\ &= 2\iiint \frac{1}{(2\pi)^2}\delta(E_2 - E_1 - E_k)\, k^2 \sin\theta\, dk\, d\theta\, d\phi \\ &= \frac{n_r^3 E_{21}^2}{\pi^2 \hbar^3 c^3} \end{aligned} \tag{1.1}$$

noting that the factor of 2 here is not for spin, as in the case of electrons and holes, but instead comes from the two allowed polarizations. Also note that we made use of the fact that $E_k = \hbar\omega = \hbar k c / n_r$. Then, we can write the photon density (number of photons per unit volume per energy) directly as

$$S(E_{21}) = N(E_{21})\, n_{ph} \tag{1.2}$$

where n_{ph} is just the number of photons.

Now we can define the transition rate per incident photon inside of a particular energy level as

$$B_{12} = \frac{2\pi}{\hbar}\left|H'_{12}\right|^2 \tag{1.3}$$

and use Fermi's Golden Rule for upward transitions to write the rate

$$\begin{aligned} R_{12} = r_{12}(E)dE &= \frac{1}{V}\sum_k B_{12}\delta(E_2 - E_1 - \hbar\omega)\, 2 n_{ph} f_1 (1 - f_2) \\ &= B_{12} f_1 (1 - f_2)\, S(E_{21}) \end{aligned} \tag{1.4}$$

Now we can use the downward transition rate to write the stimulated emission rate per unit volume as

$$R_{21}^{stim} = r_{21}^{stim}(E)dE = B_{21} f_2 (1 - f_1)\, S(E_{21}) \tag{1.5}$$

and the spontaneous emission rate per unit volume (which is independent of the photon density S) as

$$R_{21}^{spon} = r_{21}^{spon}(E)dE = A_{21} f_2 (1 - f_1) \tag{1.6}$$

Now, at thermal equilibrium, the Fermi levels are not split, so we have $F_1 = F_2$ and the total upward and downward transitions must balance each other, so

$$R_{12} = R_{21}^{stim} + R_{21}^{spon} \tag{1.7}$$

From here we see

$$\frac{A_{21}}{B_{12}\exp E_{21}/k_B T - B_{21}} = N(E_{21})\frac{1}{\exp E_{21}/k_B T - 1} \tag{1.8}$$

and compare coefficients to see that $B_{21} = B_{12}$. Therefore we see

$$N(E_{21}) = \frac{A_{21}}{B_{21}} = \frac{n_r^3 E_{21}^2}{\pi^2 \hbar^3 c^3} \tag{1.9}$$

Problem 2)

(A) FERMI-LEVELS To find the Fermi Levels for a bulk structure at zero temperature, we will start with the definition for the carrier density and express it in terms of the Fermi distribution (a step function at 0K)

$$\begin{aligned} n &= \frac{2}{V}\sum_{\vec{k}} f(E) \\ &= \int \rho_{3D}(E) f(E)\, dE \\ &= \int_{E_c}^{E_F} \rho_{3D}(E)\, dE \\ &= \frac{1}{2\pi^2}\left(\frac{2m_e^*}{\hbar^2}\right)^{3/2} \int_{E_c}^{E_F} \sqrt{E - E_c}\, dE \\ &= \frac{1}{3\pi^2}\left(\frac{2m_e^*}{\hbar^2}\right)^{3/2} (E_F - E_c)^{3/2} \end{aligned} \tag{2.1}$$

which leads to the final expression for the Fermi level for electrons (and by analog the expression for holes as well),

$$F_c = E_c + \frac{\hbar^2}{2m_e^*}\left(3\pi^2 n\right)^{2/3} \tag{2.2}$$

$$F_v = E_v + \frac{\hbar^2}{2m_h^*}\left(3\pi^2 p\right)^{2/3} \tag{2.3}$$

(B) DERIVING BULK GAIN EXPRESSION Starting from the expression for gain/absorption in terms of the Fermi levels,

$$\alpha(\hbar\omega) = \alpha_0(\hbar\omega)\left[f_v(k_0) - f_c(k_0)\right] \tag{2.4}$$

remembering that at 0K, $k_0 = \left(2m_r/\hbar^2\left(\hbar\omega - E_g\right)\right)^{1/2}$, and using the result from part (a), we can write

$$f_c(k) = \begin{cases} 1 & : E_c + \frac{\hbar^2 k_0^2}{2m_e^*} < F_c \\ 0 & : E_c + \frac{\hbar^2 k_0^2}{2m_e^*} > F_c \end{cases}$$

$$f_c(k) = \begin{cases} 1 & : \hbar\omega < E_g + \frac{m_e^*}{m_r}(F_c - E_c) \\ 0 & : \hbar\omega > E_g + \frac{m_e^*}{m_r}(F_c - E_c) \end{cases}$$

$$f_c(k) = \begin{cases} 1 & : \hbar\omega < E_g + \frac{\hbar^2}{2m_r}\left(3\pi^2 n\right)^{2/3} \\ 0 & : \hbar\omega > E_g + \frac{\hbar^2}{2m_r}\left(3\pi^2 n\right)^{2/3} \end{cases}$$

A similar expression holds for $f_v(k_0)$. Therefore, when $n = p$, we are left with the expression

$$g(\hbar\omega) = \begin{cases} \alpha_0(\hbar\omega) & : \hbar\omega < E_g + \frac{\hbar^2}{2m_r}\left(3\pi^2 n\right)^{2/3} = F_c - F_v \\ -\alpha_0(\hbar\omega) & : \hbar\omega > E_g + \frac{\hbar^2}{2m_r}\left(3\pi^2 n\right)^{2/3} = F_c - F_v \end{cases}$$

Problem 3)

(A) NUMERICAL FERMI LEVELS Now we can use the results from the previous question to proceed. We know that the quasi-Fermi levels for electrons is given by the expression

$$F_c - E_c = \frac{\hbar^2}{2m_e^*}\left(3\pi^2 n\right)^{2/3} \tag{3.1}$$

which gives a value of 170.87meV above the conduction band edge. Similarly, replacing the effective with that of the heavy holes, we see that the quasi-Fermi level for holes is 33.42meV below the valence band edge.

(B) GAIN SPECTRUM PLOT Starting from Eqn. 9.3.13 in the text, we can write the absorption spectrum as

$$\alpha(\hbar\omega) = C_0 \left|\hat{e}\cdot\vec{p}_{cv}\right|^2 \int \frac{2d^3\vec{k}}{(2\pi)^3}\delta\left(E_g + \frac{\hbar^2 k^2}{2m_r^*} - \hbar\omega\right)\left(f_v - f_c\right) \tag{3.2}$$

When $\hbar\omega$ is below $F_c - F_v$, $f_v = 0$, $f_c = 1$ and we are left with negative absorption (positive gain) after carrying out the integral. When $\hbar\omega$ surpasses the difference in Fermi levels, $f_c = 0$ and $f_v = 1$, meaning that we have positive absorption (negative gain) after carrying out the integral. For brevity, the integral is not solved here, but the expression for gain is found to be

$$g(\hbar\omega) = \begin{cases} 0 & : \hbar\omega < E_g \\ \alpha_0(\hbar\omega)\sqrt{\hbar\omega - E_g} & : E_g \le \hbar\omega \le E_g + E_{Fn} + E_{Fp} \\ -\alpha_0(\hbar\omega)\sqrt{\hbar\omega - E_g} & : E_g + E_{Fn} + E_{Fp} < \hbar\omega \end{cases}$$

where

$$\alpha_0 = C_0\left|\hat{e}\cdot\vec{p}_{cv}\right|^2 \frac{1}{2\pi^2}\left(\frac{2m_r}{\hbar^2}\right)^{3/2} \tag{3.3}$$

A plot of the gain spectrum is shown in Fig. 3.1.

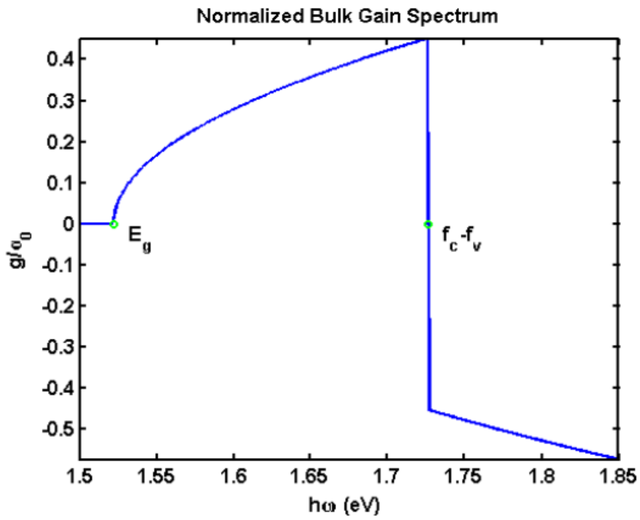

Figure 3.1: Gain spectrum for bulk GaAs